RECONSIDERING REID’S GEOMETRY OF VISIBLES

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In his ‘Inquiry’, Reid claims, against Berkeley, that there is a science of the perspectival shapes of objects (‘visible figures’): they are geometrically equivalent to shapes projected onto the surfaces of spheres. This claim should be understood as asserting that for every theorem regarding visible figures there is a corresponding theorem regarding spherical projections; the proof of the theorem regarding spherical projections can be used to construct a proof of the theorem regarding visible figures, and vice versa. I reconstruct Reid’s argument for this claim, and expose its mathematical underpinnings: it is successful, and depends on no empirical assumptions to which he was not entitled about the workings of the human eye. I also argue that, although Reid may or may not have been aware of it, the geometry of spherical projections is not the only geometry of visible figure.

I. INTRODUCTION: A QUESTION FROM BERKELEY

Encounter with the perspectival shapes of objects – the elliptical shape of the round coin lying in one’s palm, for instance – is a ubiquitous feature of visual experience. The fact of perspectival visual perception leads to an ontological question: are the perspectival shapes that we see ‘real’ features of objects? Or are they merely a useful illusion, or perhaps a shimmering world of appearances, that help us to understand the real shapes of things? Seventeenth- and eighteenth-century philosophers of perception were deeply occupied with questions of this nature, and Thomas Reid was no exception. However, Reid takes the question of the ontological status of perspectival shape to be vexed:

If it should ... be asked, To what category of beings does visible figure ... belong? I can only, in answer, give some tokens, by which those who are better acquainted with the categories, may chance to find its place.¹

But even if we think that there is no saying where to place the perspectival shapes of objects in the grand Aristotelian categories of being, we might still

think that there is some important difference in metaphysical status between perspectival and non-perspectival shapes. And we might think that we have evidence for thinking there is a difference, even if we cannot understand the metaphysical facts with the clarity needed to specify what the difference is. In fact this is what Bishop Berkeley believes. In *An Essay Towards a New Theory of Vision*, he remarks that ‘visible extension and figures are not the object of geometry’.⁴ This is not a mere curiosity, but indicates, Berkeley thinks, that even if there is no difference in ‘degree of being’ between the perspectival qualities encountered in visual experience and the qualities encountered in tactile experience – both, after all, are merely ideas – there is a distinction much like an ontological one: the objects of touch, and not of vision, are things about which it is possible to gain genuine knowledge.

If Berkeley is right about this, then there is reason to give perspectival shapes a demoted status. If they are not the sorts of things about which a science is possible, then perhaps they are nothing but *qualia*, raw feels about which nothing systematic or non-subjective can be said. Reid holds, however, that Berkeley is mistaken: the perspectival shapes of objects and their non-perspectival shapes are equally respectable; it is possible to produce a science of the second and of the first.⁵ Reid argues, however, that perspectival shapes are mathematically equivalent not to shapes drawn on planes, but instead to shapes projected onto spheres.⁶ Thus while there is an important difference between the sciences of perspectival and of non-perspectival shapes, the difference does not point to a difference in status. The study of the theorems governing the perspectival shapes, and the effort to prove those theorems, is what Reid calls ‘the geometry of visibles’.

The primary aim of this paper is reconstruction and critical evaluation of Reid’s rather obscure argument for the claim that perspectival shapes are mathematically equivalent to spherical projections. As I shall show, a large part of the reconstruction of the argument for this claim involves an elucidation of what the claim itself really amounts to. A view of the sort that

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³ The Berkeleian arguments behind Reid’s geometry of visibles are discussed in N. Daniels, *Thomas Reid’s ‘Inquiry’: the Geometry of Visibles and the Case for Realism* (Stanford UP, 1989); see esp. ch. 3. I disagree with Daniels’ interpretation of Reid’s geometry of visibles in many significant respects, as will become clear below. However, with respect to the Berkeleian background to Reid’s visible geometry, I believe that he is correct.

⁴ This fact has led Daniels (e.g., p. 12) to credit Reid with the discovery of non-Euclidean geometry much earlier than Gauss, Riemann and Lobachevsky. But Reid, as I shall argue, does not hold that the ‘visibles’ do not satisfy all of Euclid’s axioms, so he is really applying projective geometry to perspectival shape, rather than developing a genuinely non-Euclidean geometry. See Paul Wood, ‘Reid, Parallel Lines, and the Geometry of Visibles’, *Reid Studies*, 2 (1996), pp. 27–41, for an elaboration of this point.

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Reid holds can be supported by appeal to commonplaces of visual experience. When one lies on one’s back under the centre of a square ceiling, for instance, all four corners of the ceiling will appear as obtuse angles, and yet the sides of the ceiling will appear straight. What this means is that in one’s visual experience one encounters a square the angles of which add up to more than $360^\circ$. Of course a shape with these properties could never be drawn on a plane, and so it follows that the square that appears to one in this example is not geometrically planar. But examples of this sort do not establish the claim that perspectival shapes are in some way geometrically equivalent to shapes drawn on the surfaces of spheres. The square that one sees could be geometrically equivalent to a parabolic figure, for instance, or to a figure on any one of an infinite variety of regularly or irregularly curved surfaces. Reid’s argument, as will be shown here, does not depend on the empirical support provided, perhaps, by examples of this nature, although the result that he reaches is consistent with them. The argument depends only on a number of natural and appealing mathematical analyses of ordinary concepts. What makes the argument obscure is that Reid does not explicitly identify the mathematical analyses that he is using, but instead presents his argument informally. I shall uncover the details of the mathematical structure that lies behind the informal argument. My claim is that when his arguments are supplemented with some mathematical apparatus, Reid is absolutely correct: the geometry of the visible and the geometry of the spherical are one and the same. Explaining what this means, and why it is true, is the object of this paper.

II. THE GEOMETRY OF VISIBLIES

II.1. The eye as a point

Reid’s argument depends on an association of the eye with a point in three-dimensional space. But which point in space is to be appropriately identified with the eye? The eye, after all, is a complex three-dimensional object, and so occupies an infinite number of points. Although we might justify the association of the eye with a single point in space by appeal to the need for simple idealizations in any model, there is a better justification available, arising from the need to limit our discussion to objects that are in focus.
When an object is not in focus, rays of light emanating from the same point in space hit the retina in different places. This is the case, for instance, when the distance from the retina to the lens is too long or too short (see Fig. 1). However, if we assume that all of the points of the object are in focus, then associated with each point of the object is exactly one point on the retina.\(^6\) In actuality, this one-to-one correspondence between points on the surface of an object and points on the retina is exhibited only very rarely: much of the visual field, even in those who do not require corrective lenses of any kind, is not in focus at any one instant. Further, given that one’s retina, like any biological structure, is mathematically irregular (it is not, for instance, a perfect spherical section), the lens of the eye must distort in various complex ways in order to bring an object into focus, and the required distortions will vary from person to person (even holding fixed the location of the object with respect to the eye), depending on the exact shape of the individual retina. It is distortions of just the needed sort that are being accomplished by someone who keeps an object in focus as it moves closer.

To show that it is appropriate to identify the eye with a point, given that our discussion only concerns objects that are in focus, it is instructive to compare the eye with a pinhole camera, where the ‘lens’ is a hole the size of a single point, and the ‘retina’ is a flat screen onto which the light passing through the pinhole is projected. In the eye, by contrast, the retina is an irregularly curved surface, and the lens is a complex structure that fills an opening larger than a point. When we consider only objects that are in focus, however, the lens of the eye has in common with the pinhole the following property: both collect the rays emanating from a point and focus them onto a single point on the retina/screen. Therefore, given that the only relevant objects are those that are in focus, the lens of the eye is functionally equivalent to a single point in space.

Norman Daniels has claimed that Reid assumes, falsely, that the eye is a sphere, and that therefore the retina is a portion of a spherical surface, and

\(^6\) As Berkeley puts the point (\textit{NTV} §§34): ‘... any radiating point is ... distinctly seen when the rays proceeding from it are, by the refractive power of the crystalline, accurately reunited in the retina or fund of the eye: but if they are reunited, either before they arrive at the retina, or after they have passed it, then there is confused vision’.
that Reid associates the eye in his geometry of visibles with the centre of this sphere. In fact, given the justification just offered for treating the eye as a single point in space, no such assumption is required to reach the results which Reid wants. He need not make, and does not make, any empirically unjustified claims about the way in which the eye is actually constructed, in order to reach the geometrical results he is after. He is allowed to associate the eye with a single point in space because the lens of the eye collects light in just the way a single point in space collects it, if we limit our discussion to objects in focus.

II.2. Visible figure and other ‘visible’ concepts

Reid gives (Inq VI vii, p. 96) a recipe for determining the visible figure, or perspectival shape, of an object:

Objects that lie in the same right line [i.e., straight line on a plane] drawn from the centre of the eye, have the same position, however different their distances from the eye may be: but objects which lie in different right lines drawn from the eye's centre, have a different position.... Having thus defined what we mean by the position of objects with regard to the eye, it is evident that, as the real figure of a body consists in the situation of its several parts with regard to one another, so its visible figure consists in the position of its several parts with regard to the eye; and, as he that hath a distinct conception of the situation of the parts of the body with regard to one another, must have a distinct conception of its real figure; so he that conceives distinctly the position of its several parts with regard to the eye, must have a distinct conception of its visible figure.

The visible figure of an object is the conjunction of the visible positions of each of the points occupied by the object’s surface (it is difficult to define ‘surface’ for a three-dimensional object in mathematical terms, but an informal distinction between surface points and internal points can be assumed for present purposes). But what is the visible position of a point in space? Reid takes himself to have defined the notion in this passage; but in fact his definition is ambiguous between the following two claims:

7 In support, Daniels quotes the following passage: ‘I require no more knowledge in a blind man, in order to his being able to determine the visible figure of bodies, than that he can project the outline of a given body, upon a surface of a hollow sphere, whose centre is in the eye. This projection is the visible figure he wants; for it is the same figure with that which is projected upon the tunica retina in vision’ (Inq VI vii, p. 95). However, as will become clear shortly, the projection of an object’s non-perspectival shape onto any surface, whether or not it is spherical, will be a visible figure of the object. A projection onto a sphere with the eye at the centre and a projection onto the irregular surface of a retina are equally good visible figures of all of the same objects. Reid is claiming that in order to have a clear conception of an object’s visible figure, one must be able to project the object’s figure onto a surface, not that the surface must be exactly the same as the surface of a retina.

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1. The visible position of a point in space is the line passing through both the eye and the point.

2. The visible position of a point in space is any point on the line passing through both the eye and the point.

(It simplifies matters to think of the lines in both definitions as beginning at the eye and extending through the point in space, i.e., as vectors emanating from the eye, but nothing to be said depends on this simplification.) Both definitions respect the fact that each point on the line through the point and the eye occupies the same place in the visual field. Either definition would yield a definition of the term ‘visible figure’ which Reid uses in the passage: under (1), the visible figure is a set of lines; under (2) it is any one of an infinite number of different sets of points, depending on which of the infinite number of points on each of the relevant lines we select. However, we want our formal definition of visible figure to conform to the informal definition of visible figure as ‘the shape that an object appears to have’. And so our formal definition should imply that various things that we take to be true of the perspectival shapes of objects are true under our formal definition. In particular, a formal definition of visible figure is unsatisfactory if either of the following is false:

(a) Each position within the visible figure of an object is occupied by one and only one point
(b) The visible figure of an object is ‘path-connected’: from any point in the visible figure to any other there is a path that passes only through points that are also within the visible figure of the object.

Given (a), definition (1) of visible position and the accompanying definition of visible figure are unsatisfactory: if the position of a point on the surface of an object is defined as a line, then each position is occupied by an infinite number of points. Definition (2) of visible position, however, suggests that the visible figure of an object should be defined as any collection of points constructed by selecting one point of equivalent visible position for each point on the surface of the object. However, under this definition, neither (a) nor (b) need be true. Since a line from the eye will often pierce the surface of a three-dimensional object in more than one place, if we include a point in the visible figure for each surface point, then the visible figure might include points that share visible position with one another. But the import of (a) is that where we see only one point, the visible figure should contain only one point.

In order to account for (b), a further restriction on the set of points that make up the visible figure is required. If any point sharing a visible position...
with a surface point of the object could be included in the visible figure of the object, then the visible figure could be a disconnected ‘scattershot’ of points in three-dimensional space. For instance, consider four lines each of which passes through the eye and one of the four vertices of a square (see Fig. 2). Now imagine selecting one point from each line but at wildly different distances from the eye \((V_A, V_B, V_C\) and \(V_D\) in Fig. 2). If we selected a similarly disparate collection of points for each of the other lines from the eye through the other points of the square, we would end up with a collection of points no two of which were touching. But as indicated by \((b)\), this is not what we want.

For these reasons, we should accept definition \((2)\) of visible position, but we should think of the visible figure of an object as follows. A set \(VF\) of points in three-dimensional space is the visible figure of an object \(O\) relative to an eye \(E\) if and only if \((1)\) for each point on the surface of \(O\), there is a member of \(VF\) with the same visible position; \((2)\) no two members of \(VF\) share a visible position; \((3)\) the set of points is path-connected: for any two points in the set there is a path from the one to the other that passes only through other members of the set.

Under this definition, the relation ‘is the visible figure of’ is many–many. That is, many different objects will have the same visible figure, and there will be many possible visible figures for any given object. In Fig. 3, squares \(ABCD\) and \(FGHI\) have all the same visible figures; any visible figure of the one is a visible figure of the other and \textit{vice versa}. (In fact each is a visible figure of the other.) Further, the visible figure need not be a regular shape such as a square. The curvy lines connecting points \(a, b, c\) and \(d\) serve as the visible figure of either \(ABCD\)
The curve connecting \( a \) and \( b \), for instance, lies on a plane that includes \( A, B, F, G \) and the eye, and so it appears to the eye to be a straight line that shares the visible positions of the points of \( AB \) and \( FG \).

A visible figure is not an appearance, a sensation or any sort of mental state, nor would a complete characterization of the visible figure of an object need to appeal to any features of the mental life of the perceiver. A visible figure is just a set of points in three-dimensional space defined by reference to the position of an object and the position of an eye. Further, there need not be any entities occupying the points included in the visible figure of the object. To grasp the visible figure of a thing, we need not imagine that the points included in the visible figure are occupied by certain ghostly entities, nor, even less, that those points are occupied by physical things of the same sort as any ordinary physical object. The concept of a visible figure has no greater or lesser ontological significance than the concept of any other mathematically defined set of points.

Using the definition of visible figure, it is a short step to formal definitions of more specific visible figures, such as \('\text{visible line segment}'\) and \('\text{visible triangle}'\). A visible line segment is a visible figure all of the points of which lie on a plane with the eye. A visible triangle is a visible figure which is equal to the union of three visible lines, \( a, b \) and \( c \) (the \('\text{visible sides}'\) of the visible triangle), where the intersection of \( a \) and \( b \), \( a \) and \( c \), and \( b \) and \( c \) each contains exactly one point; these three points of intersection are the three \('\text{visible vertices}'\) of the visible triangle. Given additional definitions of \('\text{visible length}'\) and \('\text{visible angle}'\), we can define notions such as \('\text{visible equilateral triangle}'\) and \('\text{visible square}'\).

Reid defines visible length in the same way as Berkeley:

\[ \text{Apparent magnitude [i.e., 'visible length']} \] is measured by the angle which an object subtends at the eye. Supposing two right lines drawn from the eye to the extremities of the object, making an angle, of which the object is the subtense, the apparent magnitude is measured by this angle. This apparent magnitude is an object of sight, and not of touch. Bishop Berkeley calls it \textit{visible magnitude}.\(^8\)

(See Fig. 4 above for an illustration.) Given this definition, we would say that a visible equilateral triangle is a visible triangle all of whose visible sides have

the same visible length. A visible angle is what is sometimes referred to as a ‘dihedral angle’, or the angle between two planes that meet at a line. (Fold a piece of paper in half and hold it up so that the fold extends directly from your eye: the angle between the planes of the two halves of the paper is the visible angle.) In Fig. 5, to determine the visible angle $\angle BAC$, one considers the angle between the plane which is defined by the eye, $A$ and $C$ (‘plane $ACE$, for short), on the one hand, and plane $ABE$, on the other. This angle is equal to the planar angle between two lines each of which is perpendicular to the line connecting the eye and point $A$ (line $AE$, the ‘hinge’ of the two planes), and of which one lies on the plane $ABE$ and the other on the plane $ACE$.

Using these definitions, it is possible to prove that some, but only some, of the geometrical facts about planar figures are also true of their visible counterparts. For instance, it can be proved that all of the visible angles of a visible equilateral triangle are equal.

However, some facts about planar figures are not true of their visible counterparts. For instance, it is not the case that the visible angles of a visible triangle add up to $180^\circ$. Imagine the eye placed at the corner of a room where two walls and the ceiling meet. (See Fig. 6.) Now consider three points each one foot from the corner of the room, and each on one of the three lines that meet in the room’s corner (points $A$, $B$ and $C$ in Fig. 6). Now connect these three points with lines drawn on the two walls and the ceiling. The resulting figure is a visible triangle, but the angle between any two of the planes that meet at the room’s corner is $90^\circ$. The result is that the three visible angles are each equal to $90^\circ$, and so the visible triangle’s visible angles add up to $270^\circ$. But triangle $ABC$ is also a planar triangle, in addition to being a visible triangle. So although its visible angles add up to $270^\circ$, its planar angles add up to $180^\circ$. This is noteworthy only because it illustrates the discrepancy between the planar and visible features of single objects.
It is important that, suggestive as these results are, I have not shown (and it is not true) that visible figures lie on spheres. Given a fixed location of the eye, every object has an infinite number of visible figures that are planar. Given the definition of visible figure, we can construct a planar visible figure for every object as follows: for each point on the surface of the object, select exactly one point on a plane that intersects the line from the eye to the surface point; since there are an infinite number of such planes, there are an infinite number of planar visible figures for each object. Euclidean planar facts about these, such as that the angles of a triangle add up to 180°, do not hold true when the Euclidean concepts of an angle and a line are replaced by their visible counterparts, but this consequence does not show anything about the geometry of visible figures: visible angles, for instance, are simply different mathematical objects from planar angles, so there is no reason to expect them to behave in the same way. If a visible figure is in fact planar, then it obeys the laws of Euclidean geometry. And ‘visible’ theorems, such as, for instance, ‘The visible angles of a visible equilateral triangle are equal’, can be derived from the axioms of Euclidean geometry. In proving claims of this nature we need never deny any Euclidean axioms. I believe that Reid would deny none of this. But if not, then what does the claimed equivalence between the geometry of the visibles and spherical geometry amount to?

Imagine a function $V$ that maps sentences of geometry which do not invoke any ‘visible’ concepts onto sentences in which each non-‘visible’ concept is replaced with its corresponding ‘visible’ concept. So, for instance, $V$ maps the Pythagorean theorem, ‘In any right triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides’, to the visible Pythagorean theorem, ‘In any visible right triangle, the square of the visible length of the visible hypotenuse is equal to the sum of the squares of the visible lengths of the other two visible sides’ (a ‘visible right triangle’ has one visible right angle).9 This mapping will not always preserve truth-value. While the Pythagorean theorem is true, the visible Pythagorean theorem is false: visible triangle $ABC$ in Fig. 6 above presents a counter-example. I shall call a grammatical sentence of geometry $G$ invoking no visible concepts ‘proof-theoretically equivalent to its visible counterpart’ $V(G)$ if and only if (1) the truth-value of $G$ is the same as the truth-value of $V(G)$; and (2) if $G$ is proved (or proved false) with a set of non-visible sentences $P$, then the set of visible counterparts of the sentences in $P$ is a proof of $V(G)$, or a proof of its falsity.

9 We do not transform arithmetical concepts like ‘equals’ into corresponding visible concepts. It is difficult to draw the distinction between the geometrical concepts (like the concept of an angle), which are transformed by this mapping, and the non-geometrical concepts (like the concepts of ‘+’ and ‘=’), which are not. I shall not tackle this problem here.
Reid’s claim is that every theorem of spherical geometry – theorems that invoke the concepts (yet to be defined) of ‘spherical length’, ‘spherical angle’, etc. – is proof-theoretically equivalent to its visible counterpart. Since it is also the case that for every theorem of visible geometry there is exactly one theorem of spherical geometry that has it as a visible counterpart, it follows that there are no theorems of visible geometry for which no proof-theoretically equivalent theorem of spherical geometry cannot be found. If this claim is true, then in order to prove a theorem of visible geometry, one needs only to prove the corresponding theorem of spherical geometry. Further, by proving a theorem of visible geometry, one can reconstruct a theorem and proof of spherical geometry by working backwards and replacing all of the ‘visible’ concepts with ‘spherical’ concepts in the theorem and proof. If this is true, then the truths of spherical geometry and the truths of visible geometry are equivalent, in an important sense.

I have shown already that theorems of planar geometry do not in general satisfy the conditions of proof-theoretical equivalence with theorems of visible geometry – the visible counterparts of many true theorems of planar geometry are false. But I have not yet shown that theorems of spherical geometry are proof-theoretically equivalent to theorems of visible geometry. The next section demonstrates why this must be so; it demonstrates, that is, that spherical geometry is truly a geometry of visibles.

II.3. Reid’s argument

Reid’s argument proceeds in three stages. He shows that any sentence about a ‘spherical line’ is proof-theoretically equivalent to its visible counterpart; then he shows that this is so for any sentence about a ‘spherical angle’ made by two ‘spherical lines’; finally, he extends these results to sentences about ‘spherical triangles’.

The first stage concerns spherical lines (Inq VI ix, p. 103):

Supposing the eye placed in the centre of a sphere, every great circle of the sphere [i.e., circle drawn on the sphere concentric with it] will have the same appearance to the eye as if it was a straight line; for the curvature of the circle being turned directly toward the eye, is not perceived by it.

So far Reid is pointing out only that every great circle of a sphere centred at the eye is a visible line. Lying behind this claim is a definition of the concepts of ‘spherical line’ and ‘spherical line segment’: a spherical line is a great circle of a sphere; a spherical line segment is an arc of a great circle of

10 Daniels (p. 10) claims that Reid’s view is that ‘the geometry of visibles is consistent if spherical geometry is’. This is to invoke only clause (i) in the definition of ‘proof-theoretic equivalence’ offered in the main text above. But, as will become clear, Reid takes the equivalence between the geometry of visibles and spherical geometry to be stronger than this.
a sphere. Reid goes on to note that not every visible line is an arc of a great circle centred at the eye (Inq VI ix, p. 103):

... any line which is drawn in the plane of a great circle of the sphere, whether it be in reality straight or curve, will appear straight to the eye.

For Reid, the question of whether or not a path-connected set of points is ‘in reality straight or curve’ depends on whether the Euclidean planar notions of ‘straight’ or ‘curve’ apply to it. And so the claim made so far is merely a trivial consequence of the definition of visible figure: every spherical line is a visible line, but not vice versa. This does not establish anything like proof-theoretic equivalence between sentences concerning spherical lines, or spherical line segments, and their visible counterparts, sentences in which the spherical concepts are replaced by corresponding visible concepts. Reid attempts to establish this, or at least to illustrate it, in the next paragraph:

Every visible right line will appear to coincide with some great circle of the sphere; and the circumference of that great circle, even when it is produced until it returns into itself, will appear to be a continuation of the same visible right line, all the parts of it being visibly \textit{in directum}.

I believe that Reid is here stating that there is a proof-theoretic equivalence between a particular theorem of spherical geometry and its visible counterpart. The theorem he has in mind is ‘A spherical line has a constant spherical slope’. That this must be what Reid is saying can be shown by thinking about regular lines. In Euclidean geometry, the slope of a curve is the equation for the rate of change in position of the points of the curve with respect to one of the axes. It is a fact of Euclidean geometry that every regular line has a constant slope. Another way of putting this point is to say that the direction in which the line is heading is the same, no matter where on the line we are. All the parts of the line, that is, are \textit{in directum} – standing on any point of the line, if we were to point in the direction the line is travelling at that point, we would point at the line itself.

The ‘spherical position’ of a point is the three angles with respect to the \(x\), \(y\) and \(z\) axes of a radius drawn to the point with the centre of the sphere at the origin (see Fig. 7). Any point on this radius (line \(CA\) in Fig. 7) has the
same spherical position as any other point on this radius. A ‘spherical curve’ is any path-connected set of points each of which is equidistant from the origin. The ‘spherical slope’ of a spherical curve is the equation for the rate of change of spherical position in the points of the curve. The ‘visible slope’ of a visible curve, correspondingly, is the equation for the rate of change of visible position of the points of the curve.

What Reid is claiming in the passage just quoted is this: if a spherical curve has a constant spherical slope, then every visible figure of that spherical curve has a constant visible slope. He goes on (Inq VI ix, pp. 103–4) to explain why this must be true:

For the eye, perceiving only the position of objects with regard to itself, and not their distance, will see those points in the same visible place which have the same position with regard to the eye, how different soever their distances from it may be. Now, since a plane passing through the eye and a given visible right line, will be the plane of some great circle of the sphere, every point of the visible right line will have the same position as some point of the great circle; therefore, they will both have the same visible place, and coincide to the eye; and the whole circumference of the great circle, continued even until it returns into itself, will appear to be a continuation of the same visible right line.

Hence, it follows –

That every visible right line, when it is continued in directum, as far as it may be continued, will be represented by a great circle of a sphere, in whose centre the eye is placed.

The first sentence of this passage is easily misunderstood. Reid deduces that the eye ‘will see those points in the same visible place which have the same position with regard to the eye’ from the fact that points lying on the same line with the eye have the same visible position, regardless of their distance from the eye. But this implies that Reid must be using the phrase ‘position with regard to the eye’ to mean something different from the term ‘visible position’, otherwise no deduction would be required; that is, if the terms are being used synonymously, then it is a tautology that the eye ‘will see those points in the same visible place which have the same position with regard to the eye’. I suggest that Reid is using the term ‘position with regard to the eye’ to mean ‘spherical position’, in the sense just defined. When so understood, he can be taken to be making the following non-tautological claim: two points have the same visible position if and only if they have the same spherical position with respect to a sphere centred at the eye.11 He

11 This claim is easily proved. (a) If two points have the same visible position, then they lie on the same line drawn from the eye, and so on the same radius of a sphere drawn from the eye to the further point. But any two points on such a radius share the same spherical position. (b) If two points have the same spherical position, they lie on the same radius drawn from the eye to the farther of the two points. So they have the same visible position.
then uses this fact to show that any ‘continuation’ of the spherical line is a ‘continuation’ of the visible line, and vice versa. The slope of a curve at a point tells us in what direction – that is, towards what position – we must travel in order to continue the curve. And so Reid is claiming that whatever the spherical slope of the spherical line is at any given point, the visible line has the same visible slope, and vice versa. And thus it is shown that any theorem about spherical lines and their spherical slopes is proof-theoretically equivalent to the analogous theorem about visible lines and their visible slopes; after all, facts about lines and slopes, whether spherical or visible, are simply a function of the spherical or visible position of the points that make them up. So, given that spherical position and visible position are no different, theorems about the one and theorems about the other are true for the same reasons.

Before moving to the second stage of the argument, it is worth drawing attention to the way in which Reid phrases the conclusion of the first stage: every visible right line, when it is continued in directum, as far as it may be continued, will be represented by a great circle of a sphere, in whose centre the eye is placed. In what sense is a great circle the ‘representative’ of a visible line? If Reid had said that the great circles were representatives of the non-perspectival figure of some curve, then he would be making the point (familiar from Berkeley, *NTV* §§139–48) that visible figures of an object are a ‘sign’ of the non-perspectival figure of the object; that is, they tell us what the non-perspectival figure of the object is. But this is not what he says. He says instead that the great circles are the representatives of visible lines. Great circles are visible lines, although not all visible lines are great circles, so in what sense are they the representatives of visible lines? We might understand this ‘representation’ talk like this: one thing is a representative of another when it can speak for the other. Perhaps what Reid means is that great circles ‘speak for’ visible lines in the sense defined by proof-theoretic equivalence. That is, sentences concerning them and invoking spherical concepts (spherical positions, spherical angles, etc.) are proof-theoretically equivalent to sentences invoking visible concepts instead. To put this another way, if you want to know whether or not a visible line has a particular visible feature, and why it does or does not, all you need to do is consider if the spherical line has the analogous spherical feature. The spherical will tell you about the visible.

In the second stage of the argument, concerning spherical angles, Reid writes (*Inq VI* ix, p. 104)

the visible angle comprehended under two visible right lines, is equal to the spherical angle comprehended under the two great circles which are the representatives of these visible lines. For, since the visible lines appear to coincide with the great circles,
the visible angle comprehended under the former must be equal to the visible angle comprehended under the latter. But the visible angle comprehended under the two great circles, when seen from the centre, is of the same magnitude with the spherical angle which they really comprehend, as mathematicians know; therefore, the visible angle made by any two visible lines is equal to the spherical angle made by the two great circles of the sphere which are their representatives.

In this passage Reid invokes two different notions of an angle which I have yet to discuss. He writes of the angle that two intersecting curves ‘really comprehend’; this is what I shall call the ‘real angle’ between two intersecting curves. And he writes of the ‘spherical angle’ between two different great circles of a sphere. How are these notions defined? I suggest that by the ‘real angle’ between two curves that intersect at point A (see Fig. 8), Reid means to refer to the planar angle made by lines tangent to the two curves at that point, each lying on the plane occupied by the corresponding curve. This is consistent with Reid’s usage of the term ‘real’ to mean ‘Euclidean’ or ‘planar’. Imagine slicing a loaf of bread with two intersecting cuts. Now consider the curves on the surface of the bread made by the two cuts. Take the line tangent to the surface at the point of intersection and on the plane travelled by the knife. Do this for each cut, and consider the angle created by the two resulting lines. This is the real angle made by the two curves created by the two cuts.

The real angle between two intersecting curves is called a ‘spherical angle’ if the curves are arcs of great circles of a sphere. So, for instance, imagine replacing the bread in the above example with a perfectly spherical tennis ball, and make sure that both cuts pass through the centre of the ball. The real angle at the point of intersection of the two resulting surface curves is a spherical angle. However, while this is a sufficient condition for being a spherical angle, it is not a necessary condition, because it could be that only

Figure 8: \( L_1 \) and \( L_2 \) are the tangents at point \( A \) to \( C_1 \) and \( C_2 \), respectively, and on their respective planes. \( \alpha \) is the real angle made by \( C_1 \) and \( C_2 \), and is equal to the planar angle of \( L_1 \) and \( L_2 \).

12 If the curve is itself a straight line, then it will be used to calculate the real angle, since the tangent will not be well defined. So if both curves are straight lines, the real angle is just the planar angle between the two lines. To give a complete formal definition of ‘real angle’ we would also need to say how the real angle is to be calculated when the derivative of one or the other curve, at the point of intersection, is not defined and the curve is not a straight line. I leave this complexity aside.
a segment of each curve near the point of intersection is an arc of a great circle of a sphere. In Fig. 9, the segments $AB$ and $AC$ of curves $D_1$ and $D_2$, respectively, are arcs of great circles of a sphere centred at $E$. Beyond points $B$ and $C$, $D_1$ and $D_2$ diverge from great circles of the sphere. But the real angle made by $D_1$ and $D_2$ at $A$ is still a spherical angle. This yields the following set of necessary and sufficient conditions for a spherical angle: the real angle between curves $D_1$ and $D_2$ that intersect at point $A$ is a spherical angle if and only if there exist segments of $D_1$ and $D_2$ both of which include $A$, and both of which are arcs of great circles of a sphere.

When we interpret in the way suggested Reid’s talk of ‘real’ and ‘spherical’ angles in the difficult passage just quoted, then he appears to be trying to prove the following:

3. If the real angle made by two visible lines is a spherical angle with respect to a sphere centred at the eye, then the visible angle made by the two visible lines is equal to the real angle made by the two visible lines.

This is provable, although it is unclear if Reid has the proof in mind. Here is the proof. Referring to Fig. 10, the real angle made by the curves $AB$ and $AC$ at point $A$ is the angle between lines $L_1$ and $L_2$. If this angle is a spherical angle, then there are points $D$ and $F$, on $AB$ and $AC$ respectively, such that $AD$ and $AF$ are arcs of circles centred at $E$. This implies that $L_1$ and $L_2$ are both perpendicular to line $AE$ — a tangent at a circle is always perpendicular to a radius of the circle. But since $L_1$ lies on plane $ABE$, and $L_2$ lies on plane $ACE$, the angle between $L_1$ and $L_2$ is equal to the dihedral angle between planes $ABE$ and $ACE$. But, by definition, this is the visible angle at $A$. Q.E.D.

It is important to see that the converse of (3) is false: there are cases of curves in which the visible angle and the real angle
are the same, but neither curve has a segment that lies on a sphere centred at the eye. This is possible because, first, although lines perpendicular to the hinge between two planes make a planar angle equal to the dihedral angle between the planes, there are many other lines lying on the same planes that make a planar angle equal to the dihedral angle and that are not perpendicular to the hinge. In fact, as Fig. 11 illustrates, given two planes that intersect on a line \( AE \), and given any line through point \( A \) on one of the planes (line \( AC \) in Fig. 11), it is possible to find a line on the other plane (line \( AB \) in Fig. 11), also passing through \( A \), that makes a planar angle with the first equal to the dihedral angle between the planes. In Fig. 11, such a line was located by identifying the intersection between plane \( ABE \) and a \( 45^\circ \) cone with \( AC \) as its axis.

What this implies is that the mere fact that the visible angle made by two curves is equal to the real angle made by those curves does not imply that the curves are circular near the point of intersection: the tangents at that point need not be perpendicular to the line of sight for the real and visible angles to be the same. But if the tangents are not perpendicular to the line of sight at that point, then the curves are not coincident with circles centred at the eye. What this means is that although sentences about spherical angles are proof-theoretically equivalent to sentences about visible angles, they are not the only sentences that are. There are many pairs of curves that are not spherical lines but make real angles equal to their visible angles. In fact, given any curve at all, it is possible to construct an intersecting curve that makes a real angle and a visible angle of equal magnitude: given the tangent at the point of intersection, using the procedure above, find a corresponding line on another plane through the eye and the intersection point that makes a real angle equal to the visible angle; then pick any curve to which that line is tangent at the point of intersection. What this implies is that the geometry of spherical angles is not the only geometry of visible angles.\(^{13}\) It is not clear whether Reid appreciated this.

\(^{13}\) So Daniels (p. 11) is wrong to say ‘Projection onto no other surface [besides a sphere] preserves the properties of visible figure’. There is an infinite number of such surfaces.
point, although he may have. When summarizing his claims about the
geometry of visibles, he never, to my knowledge, says that the *only* geometry
of the visible is spherical geometry.

What remains to be shown, although I am most of the way there already,
is that theorems about spherical figures are proof-theoretically equivalent to
theorems about visible figures. Reid argues the point only for visible
triangles – he suggests that their ‘representatives’ are spherical triangles –
although he makes the point, without offering an argument for it, regarding
visible circles (see *Inq* VI ix, p. 104). But if the point applies to visible tri-
angles, it is not difficult to extend it to all other visible figures as well. Reid
offers his argument for extending the results shown so far to spherical and
visible triangles in the following passage (p. 104):

> Hence it is evident, that every visible right-lined triangle will coincide in all its parts
> with some spherical triangle. The sides of the one will appear equal to the sides of the
> other, and the angles of the one to the angles of the other, each to each: and,
> therefore, the whole of the one triangle will appear equal to the whole of the other. In
> a word, to the eye they will be one and the same, and have the same mathematical
> properties. The properties, therefore, of visible right-lined triangles, are not the same
> with the properties of plain \[ viz planar \] triangles, but are the same with those of
> spherical triangles.

The point here is quite simple: the parts of a triangle are just three line
segments and three angles; the parts of a spherical triangle are three spher-
ical line segments and three spherical angles; and the parts of a visible
triangle are three visible line segments and three visible angles. Therefore
anything that is said about a (planar, spherical or visible) triangle can be
paraphrased into a sentence that makes no mention of triangles, but
mentions only (planar, spherical or visible) sides of certain (planar, spherical
or visible) lengths, and (planar, spherical or visible) angles of certain (planar,
spherical or visible) magnitudes. But in the first two stages of the argument
it was shown that any theorem regarding spherical lines or spherical angles
was proof-theoretically equivalent to its visible counterpart. It follows that
theorems about spherical triangles are proof-theoretically equivalent to the
corresponding theorems about visible triangles. So spherical geometry is a
geometry of the visible.

III. CONCLUSION: APPEARS–IS

It is important that nowhere does Reid’s argument depend upon any claim
to the effect that perceived objects are as they appear. He is not suggesting
that all visible figures are in fact spherical. In fact, as pointed out earlier, this
is false: a projection with respect to the eye of the surface-points of an object onto any of a variety of surfaces, planar, spherical, parabolic, or even highly irregular, is a visible figure for that object. This is important for Reid’s overall philosophy of perception. Reid was, of course, a direct realist, and no direct realist can claim that any entity is by its very nature as it appears, without invoking something like ideas or sense data – special entities that are so ‘present to the mind’ as to preclude mistake about their properties. There is no reason to think that Reid takes this to be true of visible figures. In fact, given that he takes himself to be discovering many facts about visible figures that are not obvious to the normally sighted, he must think that there is a great deal about visible figures that we do not know simply by seeing them.

Reid is claiming not that the visible is the spherical, but that the geometry of the visible is the same as the geometry of the spherical. What has been shown here is that this is true, and thus that Berkeley was wrong to base his view that visible figures are less ‘real’ than the non-perspectival shapes, or in some way lower on the metaphysical ladder, on the claim that they are not the objects of a genuine, objective science. Thus, and this would have been a point of particular importance to Reid, there is no reason to think that an encounter with the perspectival shape of an object is less an encounter with a real entity than an encounter with the object’s non-perspectival shape. In both cases, we are encountering the object itself.14

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14 For comments on drafts and helpful conversation, thanks are owed to Steven Janke, Paul Zeitz, James Van Cleve and the other participants in his 2000 Summer NEH seminar on Reid, particularly Ryan Nichols and Ed Slowik. Thanks also to Sue Chan for invaluable help in creating the figures.